FURTHER MATHEMATICS

Paper 9231/11

Paper 11

General Comments

The scripts for this paper were of a generally good quality. There were a good number of high quality scripts and many showing evidence of sound learning. Work was mostly well presented by the majority of candidates. Some, however, made rather basic transcription errors, used wrong signs, or changed powers when moving from one line of working to another. Others lost marks, needlessly, by not giving exact answers, or integer coefficients, when the question asked them to do so. Most solutions were generally set out in a clear and logical order. The standard of numerical accuracy was good. Algebraic manipulation, where required, was of a high standard.

There was no evidence to suggest that candidates had any difficulty completing the paper in the time allowed. A very high proportion of scripts had substantial attempts at all eleven questions. Once again there were few misreads and few rubric infringements.

Candidates displayed a sound knowledge of a good range of the topics on the syllabus. They tackled the questions on differential equations, implicit differentiation, reduction formulae, arc length and eigenvalues confidently. The questions on proof by induction, complex numbers, polar coordinates and mean value revealed gaps in knowledge.

Comments on specific questions

Question 1

This question was tackled competently by many candidates. Weaker candidates often made errors with the summation of a constant term.

Answer: 2015 in both cases.

Question 2

Most candidates had a suitable method in order to determine the value of *k*. Sign errors were seen fairly regularly. Many candidates did not appreciate the nature of the solution in the second part of the question. Thus many only gave one point on the line of intersection of the planes, or the direction of the line. Completely correct solutions were relatively rare.

Answers: $k = -4$; $x = t$, $y = 3t - 2$, $z = \frac{7}{4}t - \frac{5}{4}$ (OE).

Question 3

There were too many candidates who were unable to state the inductive hypothesis properly and then see that the base case was given in the question. Hence two relatively easy marks were given away. There were many spurious attempts to show that $a_{k+1} - 5 > 0$, given that $a_k > 5$. Some, who were unable to complete the initial proof, fared better on the second part of the question. Weak candidates, at this level, should realise that general results cannot be proved by considering specific examples.

Question 4

In contrast, this question was done well by many candidates. The weakest candidates usually managed to obtain the first 4 marks. The algebra involved in part **(iii)** proved to be a hurdle for some. In the final part, the common errors were to use a false substitution, or to use the results already obtained, but ignore the instruction to give an answer with integer coefficients, thus forfeiting the final mark.

Answers: **(i)** $\frac{1}{9}$, **(ii)** $\frac{7}{3}$, **(iii)** $\frac{2}{9}$; $9x^3 - 21x^2 + 2x - 1 = 0$.

Question 5

In the relatively straightforward first part of this question, many candidates only scored one of the first two marks, by only finding the value of θ at the point of intersection. Others lost a mark by writing the polar coordinates the wrong way round. The sketch graphs proved to be the best source of marks in the question, but C_2 sometimes had extra points, or started at a point on the initial line, other than the pole. In the final part of the question there were many errors, which included: taking the wrong fraction of a circle, an incorrect formula for the area of a sector, wrong limits for integration and incorrect integration.

Answers: $\left(\frac{1}{\sqrt{2}}, \frac{1}{3}\pi\right)$; $\frac{1}{12}\pi + \frac{\sqrt{3}}{2} - 1$ (OE).

Question 6

This question was well done. The implicit differentiation was handled confidently, with only a very few candidates forgetting the zero on the right hand side, or having an extra $\frac{dy}{dx}$ term. The majority were able to differentiate implicitly a second time, but a substantial minority found $\frac{dy}{dx}$ explicitly and used the quotient rule.

Answer: $-\frac{1}{16}$ (maximum).

Question 7

This question was also very well done, by most candidates. The marks dropped were usually as a result of arithmetical errors, lack of brackets, which caused sign errors, or showing insufficient working in the derivation of the given answer for the reduction formula.

Answer: $I_4 = \frac{1}{2}\pi^3 - 12\pi + 24$.

Question 8

The question instructed candidates to use $\sum_{r=1}^{n} z^{2r-1}$, thus the small number that decided to employ some other method, unfortunately, did not score marks. A pleasing number, who summed the geometric progression, then saw the cancellation to make the denominator totally imaginary. This enabled the first part of the question to be done very elegantly, by equating the real parts. Those who did not see the cancellation produced longer solutions, which were often completely correct. The second part of the question caused problems, since a substantial number of candidates did not see the need to differentiate, thus they simply tried to equate imaginary parts, without success. Those who differentiated often made basic errors, with the result that there were few completely correct solutions to this question.

Question 9

The first part of this question was done well, with only a small number of candidates using the wrong formula for arc length. The algebra and integration required were well within the scope of most candidates. Only a small number were troubled by the fractional indices, so many candidates obtained the correct answer to the first part of the question. The second part caused considerable problems for many candidates, a small fraction of which did not know an appropriate formula. The main problem was working entirely with the variable *t*. Consequently, the $\frac{dx}{dt}$ factor in the numerator was missing, or the denominator was 4, rather than 32, or both. Only the best candidates were able to do this part of the question correctly.

Answers: 32.6; 1.6

Question 10

The first three requests were carried out well, by most candidates, with only occasional small inaccuracies creeping in. The initial line of algebra, required at the start of the last part of the question was not well done, or omitted. Few candidates recognized that $\mathbf{P}^{-1}\mathbf{Q}^{-1} = (\mathbf{QP})^{-1}$. A disappointing number of candidates did not then see the need to calculate the matrix product **QP**, with the result that they could not determine eigenvectors corresponding to the eigenvalues, which were frequently stated correctly. There was considerable fruitless labour, by some candidates, on this final part of the question, often going in to several pages of working.

Answers: -3; $\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\-1\\-2\end{pmatrix}$ (OE) : $\mathbf{P} = \begin{pmatrix}1 & 1 & 1\\-1 & 1 & -1\\1 & 0 & -2\end{pmatrix}$ $\mathbf{D} = \begin{pmatrix}-3 & 0 & 0\\0 & 4 & 0\\0 & 0 & 6\end{pmatrix}$; –3, 4 and 6 and $\begin{pmatrix}2\\1\\0\end{pmatrix}, \begin{pmatrix}15\\5\\3\end{pmatrix}, \begin{pmatrix}17\\7\\3\end{pmatrix}$ (OE) respectively.

Question 11 Either

The derivation of the differential equation was well done by many candidates. Those who found expressions for $\frac{dv}{dx}$ and $\frac{d^2v}{dx^2}$, rather than $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$, had a slightly less laborious task. The general solution of the *v* – *x* equation was usually found correctly. The places where marks tended to be lost were: in the calculation of the constants and by leaving the final answer as an expression for *v*, and not *y*, as requested.

Answer: $y = \left[e^{-x}\left(\frac{1}{2}\sin 2x - \cos 2x\right) + 3 + 2x - x^2\right]^{-1}$

Question 11 Or

This question was less popular than the other alternative. The solutions received for this question were mostly good attempts. In the first part, the usual approach was to use the scalar product of the vector $\overrightarrow{QP}$ with the direction vector of each of the lines l_1 and l_2, in order to find a pair of simultaneous equations. Solving these equations readily gave the required position vectors. In parts **(i)** and **(ii)**, those familiar with the vector product were able to find the area of the triangle and the volume of the tetrahedron accurately. Rather lengthier methods were employed, by some, and these resulted in arithmetical inaccuracies, although the reasoning was essentially sound.

Answers: $\mathbf{p} = \begin{pmatrix}11\\-4\\3\end{pmatrix}$, $\mathbf{q} = \begin{pmatrix}5\\-7\\1\end{pmatrix}$; $\frac{1}{2}\sqrt{2205}$ $(= 23.5)$; 122.5 (123 accepted).

FURTHER MATHEMATICS

Paper 9231/12

Paper 12

General Comments

The scripts for this paper were of a generally good quality. There were a good number of high quality scripts and many showing evidence of sound learning. Work was mostly well presented by the majority of candidates. Some, however, made rather basic transcription errors, used wrong signs, or changed powers when moving from one line of working to another. Others lost marks, needlessly, by not giving exact answers, or integer coefficients, when the question asked them to do so. Most solutions were generally set out in a clear and logical order. The standard of numerical accuracy was good. Algebraic manipulation, where required, was of a high standard.

There was no evidence to suggest that candidates had any difficulty completing the paper in the time allowed. A very high proportion of scripts had substantial attempts at all eleven questions. Once again there were few misreads and few rubric infringements.

Candidates displayed a sound knowledge of a good range of the topics on the syllabus. They tackled the questions on differential equations, implicit differentiation, reduction formulae, arc length and eigenvalues confidently. The questions on proof by induction, complex numbers, polar coordinates and mean value revealed gaps in knowledge.

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There were too many candidates who were unable to state the inductive hypothesis properly and then see that the base case was given in the question. Hence two relatively easy marks were given away. There were many spurious attempts to show that $a_{k+1} - 5 > 0$, given that $a_k > 5$. Some, who were unable to complete the initial proof, fared better on the second part of the question. Weak candidates, at this level, should realise that general results cannot be proved by considering specific examples.

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Answer: $y = \left[e^{-x}\left(\frac{1}{2}\sin 2x - \cos 2x\right) + 3 + 2x - x^2\right]^{-1}$

Question 11 Or

This question was less popular than the other alternative. The solutions received for this question were mostly good attempts. In the first part, the usual approach was to use the scalar product of the vector $\overrightarrow{QP}$ with the direction vector of each of the lines l_1 and l_2, in order to find a pair of simultaneous equations. Solving these equations readily gave the required position vectors. In parts **(i)** and **(ii)**, those familiar with the vector product were able to find the area of the triangle and the volume of the tetrahedron accurately. Rather lengthier methods were employed, by some, and these resulted in arithmetical inaccuracies, although the reasoning was essentially sound.

Answers: $\mathbf{p} = \begin{pmatrix}11\\-4\\3\end{pmatrix}$, $\mathbf{q} = \begin{pmatrix}5\\-7\\1\end{pmatrix}$; $\frac{1}{2}\sqrt{2205}\ (= 23.5)$; 122.5 (123 accepted).

FURTHER MATHEMATICS

Paper 9231/13

Paper 13

General Comments

The scripts for this paper were of a generally good quality. Work was well presented by the majority of candidates. Solutions were generally set out in a clear and logical order. There were very few poor scripts. The standard of numerical accuracy was good. Algebraic manipulation, where required, was of a high standard. However, there were a number of places in the paper where candidates did not follow the instructions in the paper.

There was no evidence to suggest that candidates had any difficulty completing the paper in the time allowed. A very high proportion of scripts had substantial attempts at all eleven questions. Once again there were few misreads and few rubric infringements.

Candidates displayed a sound knowledge of most topics on the syllabus. Candidates tackled the mathematical induction question well. There was also much good work on the reduction formula, implicit differentiation, differential equations and eigenvalues. The complex number question posed its usual difficulties for the weaker candidates and the last part of the vector question was not well understood.

Comments on specific questions

Question 1

There was rather mixed success here. A sizeable minority were unsure what ‘two pairs of equal roots’ actually meant. Few candidates began by writing down the four equations concerning the roots of a quartic equation, two of whose roots were α, and two of whose roots were β. It was disappointing to see that, even amongst those who saw that $\alpha + \beta = 0$, many were unable to make further progress. A considerable number of candidates did not give a value for q.

Answer: $q = 0$.

Question 2

This was one of the areas of the syllabus that was not well understood by a number of candidates. The appropriate formula for arc length, when the equation of the curve is given in polar form, was not well known; this made any progress impossible. Those who knew the formula frequently obtained full marks.

Answer: 1.89.

Question 3

This proof by mathematical induction was done well by many candidates. There was a small minority of candidates who established the result by the method of differences. This received no marks, because it was not what the question was testing. The only mark available to such candidates was the final mark for the sum to infinity of the series.

Answer: $\frac{1}{2}$.

Question 4

Only the most discerning candidates were able to establish the result in the first part of this question. There were a sizeable number of candidates who thought that $\tan^{-1} x$ was the same as $\frac{1}{\tan x}$, which meant that no progress could be made. Many candidates made better attempts at the second part of the question. Here difficulties arose by not considering sufficient terms, incorrect cancellation and not realising that $\tan^{-1} 1 = \frac{\pi}{4}$.

Answer: $\tan^{-1}[n+1] + \tan^{-1} n - \frac{1}{4}\pi$.

Question 5

There were variable levels of success in establishing the reduction formula. Despite the instructions in the question, weaker candidates thought that integration by parts was required here. A number of candidates forfeited the fifth mark. Since the answer was given, a small amount of working needed to be shown. Most candidates made a good attempt to use the reduction formula and many obtained the correct result.

Answer: $-\frac{152}{105}$.

Question 6

Some weak candidates made a poor start to this question by confusing a binomial series with a geometric progression. Most, however, were able to obtain the first three marks by identifying the real part of the binomial expansion. Problems then arose for a considerable number of candidates, who could not recall the appropriate formulae for expressing the trigonometric terms in half angle form. Others' progress ground to a halt, because they could not simplify their work. In the final part, many solutions were spoiled by the inclusion of an extraneous term of 1, or by having $\sin^n\left(\frac{1}{2}\theta\right)$ when $\cos^n\left(\frac{1}{2}\theta\right)$ was required.

Answer: $2^n \cos^n\left(\frac{1}{2}\theta\right)\sin\left(\frac{1}{2}n\theta\right)$.

Question 7

This question was quite well attempted. Most candidates were able to score many of the marks available. The most common omission was not specifying the values for the second derivative, thus losing the last two marks. Errors also arose through poor notation, at this level things should be expressed more rigorously.

Answers: (2, – 2) (maximum), (–2, 2) (minimum).

Question 8

The first two parts of this question were generally well done, with only the occasional careless slip, such as finding (2**i** + **j** – 2**k**) × **n**. Very few candidates followed the instruction, given in the question, to 'use **w**'. The geometry of the situation in this final part was not appreciated. The vector product of **w** with **n** immediately gives the direction of the line *PQ,* thus enabling the vector equation of the line *PQ* to be written down, using the known point, which was found in the first part of the question. Those who produced a successful 'otherwise' solution obtained three of the available four marks. Those who did not have an equation, by omitting '**r** =' lost one mark. Once again the form of the answer was specified on the question paper.

Answers: (1, – 1,4), $\begin{pmatrix}1\\2\\1\end{pmatrix}$, $\begin{pmatrix}-5\\4\\-3\end{pmatrix}$, $\mathbf{r} = \begin{pmatrix}1\\-1\\4\end{pmatrix} + \mu\begin{pmatrix}5\\1\\-7\end{pmatrix}$.

Question 9

This question was well done and many candidates obtained full marks. The most common errors included having x for t in the complementary function, sign errors and other slips in applying the initial conditions, in order to evaluate the constants.

Answer: $x = e^{5t} + 2e^{-2t} + 0.3 \cos t - 0.1 \sin t$.

Question 10

This was a further question where marks were lost by not reading the instructions in the question carefully enough. Most candidates' algebra was sufficiently good to earn the first three marks. The question then said 'Hence', so those who proceeded to use the discriminant to establish the result lost one mark. Likewise, those who used differentiation, in order to find the turning points, also lost a mark. A surprising number of candidates wrote down the correct equation of the asymptote of C, but did not include it anywhere on their graph. Further marks were lost in the final part, by not including the coordinates asked for in the question.

Answers: $\left(-3, \frac{9}{2}\right)$, $\left(\frac{1}{3}, -\frac{1}{2}\right)$; $y = 4$; $(0,0)$, $\left(\frac{3}{4}, 0\right)$, $\left(-\frac{4}{3}, 4\right)$.

Question 11 Either

The row reduction of the matrix **M** to echelon form was either incomplete, or contained arithmetical errors. A substantial number of candidates did not realise that they had to prove that the given vectors were linearly independent. Merely stating the result only obtained the fifth of the five marks available in part **(ii)**. Most gave a correct statement in part **(iii)**. Further marks were lost in the row reduction of the augmented matrix in part **(iv)**. Conclusions were often too vague and showed confusion as to which vectors were in V and which were in W.

Answers: **(i)** 3; **(iii)** W is not a vector space as it does not contain the zero vector.

Question 11 Or

This question was, by far, the more popular of the two alternatives. The value of α was invariably obtained correctly in the initial request. Most candidates were then able to use the value obtained to form and solve the characteristic equation, thus finding a further two eigenvalues. A small number of candidates did not identify λ_1 and λ_2 correctly, from the information given in the question. In part **(ii)** it was not uncommon for candidates to omit finding the eigenvector corresponding to the eigenvalue of – 9, despite the question asking for all three eigenvectors. Arithmetical accuracy was sometimes deficient in this part of the question. The final part of the question frequently lacked rigour, with steps in the argument being omitted. The final mark was sometimes lost, by not inserting the values of the eigenvalues, that had been found earlier.

Answers: $\alpha = -1$; **(i)** $\lambda_1 = 6$ and $\lambda_2 = 3$, **(ii)** $\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$, $\begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ (OE) for $\lambda = -9$, 6 and 3 respectively.

FURTHER MATHEMATICS

Paper 9231/21

Paper 21

Key messages

To score full marks in the paper candidates must be well versed in both Mechanics and Statistics, though any preference between these two areas can be exercised in the choice of the final optional question.

All steps of the argument or derivation should be written down in those questions which require a given result to be shown. Although less detail may suffice in questions where an unknown value or result is to be found, candidates are advised to show sufficient working so that credit may be earned for their method of solution even if an error occurs.

Non-standard symbols introduced by candidates should be defined or explained, for example by showing forces or velocities in a diagram within their written answers. Annotating a diagram in the question paper is insufficient, since this will not be seen by the Examiners.

General comments

Almost all candidates attempted all the questions, and while very good answers were frequently seen, the paper discriminated well between different levels of ability. In the only question which offered a choice, namely **Question 11**, there was a marked preference for the Statistics option, though some of the candidates who chose the Mechanics option produced good attempts. Indeed all questions were answered well by some candidates, with **Questions 2, 4, 9** and **10** found to be the most challenging.

Advice to candidates in previous reports to set out their work clearly, with any corrections legible and the replacements to deleted attempts readily identifiable was seemingly heeded by many. When relevant in Mechanics questions it is helpful to include diagrams which show, for example, what forces are acting and also their directions as in **Question 4** and the directions of motion of particles, as in **Question 5**. Where the meaning of symbols introduced by candidates is not clearly defined in this way or is not otherwise obvious, it is helpful to include a definition or explanation. This is also true in Statistics questions, and thus in **Questions 8** and **10** any symbols used when stating the hypotheses should either be in standard notation or should be correctly defined.

The rubric for the paper specifies that non-exact numerical answers be given to 3 significant figures, and while most candidates abided by this requirement, a few rounded intermediate results to this same or lower accuracy with consequent risk that the final result is in error in the third significant figure. Such premature approximations should therefore be avoided, particularly when the difference of two numbers of similar sizes is to be taken, as can happen when taking the difference of means or estimating variances for example.

Comments on specific questions

Question 1

Most candidates found the value 2·5 of T correctly by differentiating the given expression for P's velocity and then solving $4T - 4 = 6$. Somewhat fewer then evaluated the velocity v using this value of T and found the magnitude of the radial component of the acceleration of P from $\frac{v^2}{0.25}$. Candidates should be aware that the references in the question to the velocity and acceleration of P are to its linear velocity and acceleration, and there is no need here to consider its angular velocity or acceleration.

Answer: 121 m s^{-2}.

Question 2

Since the motion is stated to be simple harmonic, the value $\frac{1}{4}$ of the parameter ω in the standard SHM equation $\frac{d^2x}{dt^2} = -\omega^2 x$ may be readily found from $0{\cdot}625 = \omega^2 \times 10$, and hence the period T from $\frac{2\pi}{\omega}$. Many candidates also went on to successfully apply another standard SHM equation, namely $v^2 = \omega^2 (a^2 - x^2)$, to find the amplitude a of the motion. The final part, however, seemed to present more of a challenge. When attempting such a problem concerning SHM, it may be advisable to consider carefully whether to relate the displacement x to the time t by $x = a \sin \omega t$ or $x = a \cos \omega t$. While both approaches are valid if employed correctly, one may be more appropriate than the other, involving less work or being less susceptible to error. The deciding factor is usually the value of x when $t = 0$. Turning to the final part of this question, it is perhaps easiest to find the times t_{OC} from C to O (which equals that from O directly to C) and t_{OM} from O to M respectively from $10 = a \sin \omega t_{CO}$ and $\frac{1}{2} a = a \sin \omega t_{OM}$. The time from C to M is then $t_{CO} + t_{OM}$, with the latter time incidentally being $\frac{2\pi}{3}$ s. The popular alternative of the same method with sin replaced by cos is potentially correct, but care must be taken since $t = 0$ corresponds to either A or B, and the working must therefore be appropriately modified. It is not of course valid to argue that because M is the mid-point of OB then the time from O to M is one-eighth of the period, namely π s, since the particle's speed is not constant.

Answers: **(i)** 8π s; **(ii)** 26 m; 3·67 s.

Question 3

Applying Newton's second law of motion to the particle P in a radial direction at the point where OP makes an angle θ with the upward vertical gives an equation for the reaction R involving P's unknown speed v there. A second equation for v results from conservation of energy, allowing its elimination to show that R satisfies the given equation. Taking the reaction to be zero when P loses contact with the sphere and replacing v by $2u$ enables $\cos \theta$ to be either found (with value $\frac{8}{11}$) or eliminated from any two of the earlier equations, in either case giving u in terms of a and g. Most candidates seemed to experience little difficulty here.

Answer: $\sqrt{\left(\frac{2ag}{11}\right)}$.

Question 4

Most candidates were sufficiently familiar with Hooke's law to realise that it could be used to find the required modulus of elasticity λ of the string once the tension T in the string was known, since here $T = \lambda \left(\frac{\left(\frac{7a}{5}\right)}{\left(\frac{3a}{5}\right)}\right) = \frac{7\lambda}{3}$. T may be most easily found to be $\frac{7W}{2}$ by taking moments for the rod about B, since the only two other forces involved are the known weights of the rod and particle. The popular alternative of equating the vertical component of T to the sum of these two weights is of course invalid, since account must also be taken of the unknown force acting on the rod at B. Finding this in the second part of the question was usually done by finding and then combining two perpendicular components of the force, almost always in the horizontal and vertical directions though other choices are equally valid. Candidates should note that a direction is not defined by simply stating an angle; for completeness they must also state which direction the angle is measured from.

Answers: **(i)** $\frac{3W}{2}$; **(ii)** $\left(\frac{\sqrt{57}}{2}\right) W$ at an upward angle from AB produced of 36·6°.

Question 5

Almost all candidates were able to formulate two equations for the speeds v_A and v_B of the spheres A and B after the first collision by means of conservation of momentum and Newton's restitution equation. These simultaneous equations were usually solved correctly to give the speed of B in terms of e, which is needed to formulate analogous equations for the speeds of B and C after the second collision. Solving these in turn yields the given final speed v_C of C, and the required corresponding speed v_B' of B, though a significant number of candidates overlooked the latter at this point. For the three spheres to have the same final momentum implies that $3v_A = 2v_B' = v_C$, and substitution of the speeds into any two of these three equations yields the required values of e and e'. Those candidates who chose to involve instead the impulses experienced by the spheres were invariably unsuccessful, and this approach is inadvisable.

Answer: $(1 + e)(2 - e')\frac{u}{5}$; $e = \frac{2}{3}$, $e' = \frac{1}{2}$.

Question 6

Those candidates who recalled the most appropriate form of the expression for the pooled (or two-sample) estimate of a common variance needed only to substitute the given information in order to produce the equation $\frac{\left(11 - \frac{5^2}{N} + 160 - \frac{10^2}{10}\right)}{(N + 10 - 2)} = 12$. For those who did not, an equivalent form is given in the *List of Formulae*, though this also needs the general definition of a mean x of n values x_i, namely $\bar{x} = \sum \frac{x_i}{n}$.

Unfamiliarity with these formulae made successful progress more difficult, though still quite possible provided candidates did not confuse biased and unbiased estimates of population variance. The concluding step is to rearrange this equation as a quadratic in N, and accept the solution which is an integer.

Answer: 5.

Question 7

The quickest way to find Σx is to recall that the mid-point of the given confidence interval has been found from the sample mean, so this value 1·6 need only be multiplied by the sample size 8 to give the required value of Σx. The half-length 0·43 of the interval has been calculated from the unbiased estimate s^2 for the population variance so the latter can be found by equating 0·43 to $\frac{2.365s}{\sqrt{8}}$, giving $s = 0{\cdot}5143$. This approach is of course equivalent to relating 1·17 and 2·03 to $\mu \pm \frac{2.365s}{\sqrt{8}}$ and solving the resulting simultaneous equations, but some candidates would probably have found the former method quicker and less error-prone.

Finally Σx^2 is found from $s^2 = \left\{\sum x^2 - \frac{\left(\sum x\right)^2}{8}\right\}7$. When doing so, candidates must take care to use 7 and 8 appropriately in this expression, consistent with the confidence interval being based on an unbiased variance estimate.

Answer: 12·8, 22·3.

Question 8

The value of the product moment correlation coefficient r was usually found correctly as the square root of the product of the two gradients 0·38 and 0·96, though the square root was sometimes overlooked. Most candidates went on to state the null and alternative hypotheses correctly, which should be in the form $\rho = 0$ and $\rho > 0$, though some wrongly stated them in terms of r which relates of course to the sample and not the population. Comparison of the previously calculated value of r with the tabular value 0·549 leads to a conclusion of there being evidence of positive correlation. The final part requires candidates to consult the table of critical values for the product moment correlation coefficient, and in particular the values for a two-tail significance level of 5%. Since the given value of 0·507 falls between the critical values 0·497 for $n = 16$ and 0·514 for $n = 15$, it follows that the required least possible value of n is 16. In such tests of correlation, candidates should consider carefully whether one-tail or two-tail significance values are appropriate.

Answers: **(a) (i)** 0·604; **(b)** 16.

Question 9

In the challenging first part of this question, candidates should state explicitly that the probability of there being no flaws in a length x m of cloth is equal to the probability of the distance between two successive flaws being greater than x. The working of some candidates who utilised this equality was marred by using the symbol X to denote both the distance between two successive flaws, as it is defined in the question, and the number of flaws in a particular length of cloth. Other candidates appeared to believe that the required expression for $P(X > x)$ is so obvious as to require no real argument, or that it is sufficient to effectively state that X must have a negative exponential distribution, but neither of these attempts is adequate. The probability that there is at least one flaw in a 4 metre length of cloth is found from $1 - e^{-0.8x}$ with $x = 4$, though both here and in the distribution function of X the "1 –" was sometimes omitted, particularly when the latter was simply stated rather than found from $P(X \le x)$. Most candidates were aware that the probability density function is found by differentiating the distribution function, though it was not uncommon to see these two functions confused with one another. While it was generally appreciated that the upper and lower quartile values may be found by equating the distribution function to $\frac{3}{4}$ and $\frac{1}{4}$ respectively, it was less often known that the interquartile range is defined to be the difference between them.

Answers: **(ii)** 0·959; **(iii) (a)** 0 ($x < 0$), $1 - e^{-0.8x}$ ($x \ge 0$); **(b)** 0 ($x < 0$), $0.8\,e^{-0.8x}$ ($x \ge 0$); **(c)** 1·37.

Question 10

The gradient of the required regression line is first found using the standard formula, and then used with the means of the given sample values of x and y to effectively find the constant term and hence the value of y when $x = 7$. These calculations were generally performed well, though candidates should retain at least 3 significant figures in the intermediate results in order to ensure three-figure accuracy in the final result. The more obvious choice of regression line is here y on x, since the distance thrown by a child at the end of a year would seem to depend on the distance thrown one year earlier rather than the reverse. The key to the second part is the choice of test, which should here be a paired-sample test. Those candidates who made this choice usually conducted the test well, but there were many who used instead an inappropriate test. After stating the hypotheses, the first step in the calculation is to find the differences between the pairs of observations and base the test on them. It is of course essential in general to retain the signs of the differences and not just consider their magnitudes, but in this case they are all of the same sign. The mean of the resulting sample is then $\frac{6.1}{10}$ and the unbiased estimate of the population variance is 0·1632. This gives a calculated value for t of 1·64, and comparison with the tabulated value 1·833 leads to a conclusion that there is insufficient evidence to support the teacher's suspicion. As in all such tests, candidates should state their hypotheses explicitly in terms of the population and not the sample means.

Answer: 7·47 m.

Question 11 (Mechanics)

This optional question was attempted by a minority of the candidates, and the last part in particular proved challenging for many of them. Finding the required moment of inertia in the first part presented most candidates with little difficulty, requiring the use of standard formulae and usually the parallel axes theorem to formulate and then sum the individual moments of inertia of the disc, ring and rods. The latter can be treated as two pairs of collinear rods, though most candidates chose instead to find and then sum the moments of inertia of individual rods. As in all such questions where a given result must be shown, candidates should include sufficient details of their solution so as to justify full credit. Candidates who simply write down a sum of terms with no explanation whatever run considerable risk, since an error in only one term can cast doubt on the validity of their whole process and thereby lose considerable credit. To find the moment of inertia of the object about the specified axis l at A, the perpendicular axes theorem is first applied at O, thus halving the previously-found moment of inertia, and then the parallel axes theorem is applied to translate from O to A. The many candidates who instead performed these two operations in reverse order should realise that this is invalid, since the moments of inertia about two axes in the plane at A which are parallel and perpendicular to AC are not equal, unlike in the symmetric situation at O. The addition of the particle at C in the final part further changes the moment of inertia of the combined object, to $220{\cdot}5\ ma^2$. Conservation of energy then produces an equation for the initial angular speed ω, and hence u since $u = 3a\omega$. The kinetic energy is of course rotational and not linear here, and care must be taken over the potential energy since the particle at C is $6a$ from the axis at A whereas the centre of mass of the remainder of the object is $3a$ from the axis. Candidates should not assume simple harmonic motion is involved here as the angle through which the object is displaced is far from small.

Answer: $112{\cdot}5\ ma^2$; $0.35\sqrt{ag}$.

Question 11 (Statistics)

The appropriate type of distribution was usually identified correctly as being geometric, and the expected values were then usually found correctly from $200pq^{x-1}$ with $p = 0{\cdot}6$ and $q = 0{\cdot}4$, except that some candidates evaluated this expression with $x = 7$ to find the final value instead of subtracting all the previous expected values from the required total of 200. In some circumstances this error would not be significant, but unfortunately it here results in the last four rather than the last three cells needing to be combined to ensure a combined expected value of at least 5. Apart from this, the goodness of fit test was also often carried out well. Comparison of the calculated value 4·23 of χ^2 with the critical value 9·488 leads to acceptance of the null hypothesis, namely that the geometric distribution does fit the data. The final part was found to be more challenging by many candidates. The probability $p = 1 - 0{\cdot}75^5$ of obtaining at least one 6 on 5 throws of any one die is first determined, and the required result is then ${}^{10}C_4\, p^4\,(1 - p)^6$.

Answer: 0·0127.

FURTHER MATHEMATICS

Paper 9231/22

Paper 22

Key messages

To score full marks in the paper candidates must be well versed in both Mechanics and Statistics, though any preference between these two areas can be exercised in the choice of the final optional question.

All steps of the argument or derivation should be written down in those questions which require a given result to be shown. Although less detail may suffice in questions where an unknown value or result is to be found, candidates are advised to show sufficient working so that credit may be earned for their method of solution even if an error occurs.

Non-standard symbols introduced by candidates should be defined or explained, for example by showing forces or velocities in a diagram within their written answers. Annotating a diagram in the question paper is insufficient, since this will not be seen by the Examiners.

General comments

Almost all candidates attempted all the questions, and while very good answers were frequently seen, the paper discriminated well between different levels of ability. In the only question which offered a choice, namely **Question 11**, there was a marked preference for the Statistics option, though some of the candidates who chose the Mechanics option produced good attempts. Indeed all questions were answered well by some candidates, with **Questions 2, 4, 9** and **10** found to be the most challenging.

Advice to candidates in previous reports to set out their work clearly, with any corrections legible and the replacements to deleted attempts readily identifiable was seemingly heeded by many. When relevant in Mechanics questions it is helpful to include diagrams which show, for example, what forces are acting and also their directions as in **Question 4** and the directions of motion of particles, as in **Question 5**. Where the meaning of symbols introduced by candidates is not clearly defined in this way or is not otherwise obvious, it is helpful to include a definition or explanation. This is also true in Statistics questions, and thus in **Questions 8** and **10** any symbols used when stating the hypotheses should either be in standard notation or should be correctly defined.

The rubric for the paper specifies that non-exact numerical answers be given to 3 significant figures, and while most candidates abided by this requirement, a few rounded intermediate results to this same or lower accuracy with consequent risk that the final result is in error in the third significant figure. Such premature approximations should therefore be avoided, particularly when the difference of two numbers of similar sizes is to be taken, as can happen when taking the difference of means or estimating variances for example.

Comments on specific questions

Question 1

Most candidates found the value 2·5 of T correctly by differentiating the given expression for P's velocity and then solving $4T - 4 = 6$. Somewhat fewer then evaluated the velocity v using this value of T and found the magnitude of the radial component of the acceleration of P from $\frac{v^2}{0.25}$. Candidates should be aware that the references in the question to the velocity and acceleration of P are to its linear velocity and acceleration, and there is no need here to consider its angular velocity or acceleration.

Answer: 121 m s^{-2}.

Question 2

Since the motion is stated to be simple harmonic, the value $\frac{1}{4}$ of the parameter ω in the standard SHM equation $\frac{d^2x}{dt^2} = -\omega^2 x$ may be readily found from $0{\cdot}625 = \omega^2 \times 10$, and hence the period T from $\frac{2\pi}{\omega}$. Many candidates also went on to successfully apply another standard SHM equation, namely $v^2 = \omega^2 (a^2 - x^2)$, to find the amplitude a of the motion. The final part, however, seemed to present more of a challenge. When attempting such a problem concerning SHM, it may be advisable to consider carefully whether to relate the displacement x to the time t by $x = a \sin \omega t$ or $x = a \cos \omega t$. While both approaches are valid if employed correctly, one may be more appropriate than the other, involving less work or being less susceptible to error. The deciding factor is usually the value of x when $t = 0$. Turning to the final part of this question, it is perhaps easiest to find the times t_{OC} from C to O (which equals that from O directly to C) and t_{OM} from O to M respectively from $10 = a \sin \omega t_{CO}$ and $\frac{1}{2}a = a \sin \omega t_{OM}$. The time from C to M is then $t_{CO} + t_{OM}$, with the latter time incidentally being $\frac{2\pi}{3}$ s. The popular alternative of the same method with sin replaced by cos is potentially correct, but care must be taken since $t = 0$ corresponds to either A or B, and the working must therefore be appropriately modified. It is not of course valid to argue that because M is the mid-point of OB then the time from O to M is one-eighth of the period, namely π s, since the particle's speed is not constant.

Answers: **(i)** 8π s; **(ii)** 26 m; 3·67 s.

Question 3

Applying Newton's second law of motion to the particle P in a radial direction at the point where OP makes an angle θ with the upward vertical gives an equation for the reaction R involving P's unknown speed v there. A second equation for v results from conservation of energy, allowing its elimination to show that R satisfies the given equation. Taking the reaction to be zero when P loses contact with the sphere and replacing v by $2u$ enables $\cos\theta$ to be either found (with value $\frac{8}{11}$) or eliminated from any two of the earlier equations, in either case giving u in terms of a and g. Most candidates seemed to experience little difficulty here.

Answer: $\sqrt{\left(\frac{2ag}{11}\right)}$.

Question 4

Most candidates were sufficiently familiar with Hooke's law to realise that it could be used to find the required modulus of elasticity λ of the string once the tension T in the string was known, since here $T = \lambda\left(\frac{\left(\frac{7a}{5}\right)}{\left(\frac{3a}{5}\right)}\right) = \frac{7\lambda}{3}$. T may be most easily found to be $\frac{7W}{2}$ by taking moments for the rod about B, since the only two other forces involved are the known weights of the rod and particle. The popular alternative of equating the vertical component of T to the sum of these two weights is of course invalid, since account must also be taken of the unknown force acting on the rod at B. Finding this in the second part of the question was usually done by finding and then combining two perpendicular components of the force, almost always in the horizontal and vertical directions though other choices are equally valid. Candidates should note that a direction is not defined by simply stating an angle; for completeness they must also state which direction the angle is measured from.

Answers: **(i)** $\frac{3W}{2}$; **(ii)** $\left(\frac{\sqrt{57}}{2}\right)W$ at an upward angle from AB produced of 36·6°.

Question 5

Almost all candidates were able to formulate two equations for the speeds v_A and v_B of the spheres A and B after the first collision by means of conservation of momentum and Newton's restitution equation. These simultaneous equations were usually solved correctly to give the speed of B in terms of e, which is needed to formulate analogous equations for the speeds of B and C after the second collision. Solving these in turn yields the given final speed v_C of C, and the required corresponding speed v_B' of B, though a significant number of candidates overlooked the latter at this point. For the three spheres to have the same final momentum implies that $3v_A = 2v_B' = v_C$, and substitution of the speeds into any two of these three equations yields the required values of e and e'. Those candidates who chose to involve instead the impulses experienced by the spheres were invariably unsuccessful, and this approach is inadvisable.

Answer: $(1 + e)(2 - e')\frac{u}{5}$; $e = \frac{2}{3}$, $e' = \frac{1}{2}$.

Question 6

Those candidates who recalled the most appropriate form of the expression for the pooled (or two-sample) estimate of a common variance needed only to substitute the given information in order to produce the equation $\dfrac{\left(11 - \dfrac{5^2}{N} + 160 - \dfrac{10^2}{10}\right)}{(N + 10 - 2)} = 12$. For those who did not, an equivalent form is given in the *List of Formulae*, though this also needs the general definition of a mean x of n values x_i, namely $\bar{x} = \sum \frac{x_i}{n}$.

Unfamiliarity with these formulae made successful progress more difficult, though still quite possible provided candidates did not confuse biased and unbiased estimates of population variance. The concluding step is to rearrange this equation as a quadratic in N, and accept the solution which is an integer.

Answer: 5.

Question 7

The quickest way to find Σx is to recall that the mid-point of the given confidence interval has been found from the sample mean, so this value 1·6 need only be multiplied by the sample size 8 to give the required value of Σx. The half-length 0·43 of the interval has been calculated from the unbiased estimate s^2 for the population variance so the latter can be found by equating 0·43 to $\frac{2.365s}{\sqrt{8}}$, giving $s = 0{\cdot}5143$. This approach is of course equivalent to relating 1·17 and 2·03 to $\mu \pm \frac{2.365s}{\sqrt{8}}$ and solving the resulting simultaneous equations, but some candidates would probably have found the former method quicker and less error-prone.

Finally Σx^2 is found from $s^2 = \left\{\sum x^2 - \dfrac{\left(\sum x\right)^2}{8}\right\} 7$. When doing so, candidates must take care to use 7 and 8 appropriately in this expression, consistent with the confidence interval being based on an unbiased variance estimate.

Answer: 12·8, 22·3.

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Answers: **(a) (i)** 0·604; **(b)** 16.

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In the challenging first part of this question, candidates should state explicitly that the probability of there being no flaws in a length x m of cloth is equal to the probability of the distance between two successive flaws being greater than x. The working of some candidates who utilised this equality was marred by using the symbol X to denote both the distance between two successive flaws, as it is defined in the question, and the number of flaws in a particular length of cloth. Other candidates appeared to believe that the required expression for $P(X > x)$ is so obvious as to require no real argument, or that it is sufficient to effectively state that X must have a negative exponential distribution, but neither of these attempts is adequate. The probability that there is at least one flaw in a 4 metre length of cloth is found from $1 - e^{-0.8x}$ with $x = 4$, though both here and in the distribution function of X the "1 –" was sometimes omitted, particularly when the latter was simply stated rather than found from $P(X \le x)$. Most candidates were aware that the probability density function is found by differentiating the distribution function, though it was not uncommon to see these two functions confused with one another. While it was generally appreciated that the upper and lower quartile values may be found by equating the distribution function to $\frac{3}{4}$ and $\frac{1}{4}$ respectively, it was less often known that the interquartile range is defined to be the difference between them.

Answers: **(ii)** 0·959; **(iii) (a)** 0 ($x < 0$), $1 - e^{-0.8x}$ ($x \ge 0$); **(b)** 0 ($x < 0$), $0.8\, e^{-0.8x}$ ($x \ge 0$); **(c)** 1·37.

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The gradient of the required regression line is first found using the standard formula, and then used with the means of the given sample values of x and y to effectively find the constant term and hence the value of y when $x = 7$. These calculations were generally performed well, though candidates should retain at least 3 significant figures in the intermediate results in order to ensure three-figure accuracy in the final result. The more obvious choice of regression line is here y on x, since the distance thrown by a child at the end of a year would seem to depend on the distance thrown one year earlier rather than the reverse. The key to the second part is the choice of test, which should here be a paired-sample test. Those candidates who made this choice usually conducted the test well, but there were many who used instead an inappropriate test. After stating the hypotheses, the first step in the calculation is to find the differences between the pairs of observations and base the test on them. It is of course essential in general to retain the signs of the differences and not just consider their magnitudes, but in this case they are all of the same sign. The mean of the resulting sample is then $\frac{6.1}{10}$ and the unbiased estimate of the population variance is 0·1632. This gives a calculated value for t of 1·64, and comparison with the tabulated value 1·833 leads to a conclusion that there is insufficient evidence to support the teacher's suspicion. As in all such tests, candidates should state their hypotheses explicitly in terms of the population and not the sample means.

Answer: 7·47 m.

Question 11 (Mechanics)

This optional question was attempted by a minority of the candidates, and the last part in particular proved challenging for many of them. Finding the required moment of inertia in the first part presented most candidates with little difficulty, requiring the use of standard formulae and usually the parallel axes theorem to formulate and then sum the individual moments of inertia of the disc, ring and rods. The latter can be treated as two pairs of collinear rods, though most candidates chose instead to find and then sum the moments of inertia of individual rods. As in all such questions where a given result must be shown, candidates should include sufficient details of their solution so as to justify full credit. Candidates who simply write down a sum of terms with no explanation whatever run considerable risk, since an error in only one term can cast doubt on the validity of their whole process and thereby lose considerable credit. To find the moment of inertia of the object about the specified axis *l* at *A*, the perpendicular axes theorem is first applied at *O*, thus halving the previously-found moment of inertia, and then the parallel axes theorem is applied to translate from *O* to *A*. The many candidates who instead performed these two operations in reverse order should realise that this is invalid, since the moments of inertia about two axes in the plane at *A* which are parallel and perpendicular to *AC* are not equal, unlike in the symmetric situation at *O*. The addition of the particle at *C* in the final part further changes the moment of inertia of the combined object, to $220{\cdot}5\,ma^2$. Conservation of energy then produces an equation for the initial angular speed ω, and hence u since $u = 3a\omega$. The kinetic energy is of course rotational and not linear here, and care must be taken over the potential energy since the particle at *C* is $6a$ from the axis at *A* whereas the centre of mass of the remainder of the object is $3a$ from the axis. Candidates should not assume simple harmonic motion is involved here as the angle through which the object is displaced is far from small.

Answer: $112{\cdot}5\,ma^2$; $0.35\sqrt{ag}$.

Question 11 (Statistics)

The appropriate type of distribution was usually identified correctly as being geometric, and the expected values were then usually found correctly from $200pq^{x-1}$ with $p = 0{\cdot}6$ and $q = 0{\cdot}4$, except that some candidates evaluated this expression with $x = 7$ to find the final value instead of subtracting all the previous expected values from the required total of 200. In some circumstances this error would not be significant, but unfortunately it here results in the last four rather than the last three cells needing to be combined to ensure a combined expected value of at least 5. Apart from this, the goodness of fit test was also often carried out well. Comparison of the calculated value 4·23 of χ^2 with the critical value 9·488 leads to acceptance of the null hypothesis, namely that the geometric distribution does fit the data. The final part was found to be more challenging by many candidates. The probability $p = 1 - 0{\cdot}75^5$ of obtaining at least one 6 on 5 throws of any one die is first determined, and the required result is then ${}^{10}C_4\, p^4\,(1 - p)^6$.

Answer: 0·0127.

FURTHER MATHEMATICS

Paper 9231/23
Paper 23

Key messages

To score full marks in the paper candidates must be well versed in both Mechanics and Statistics, though any preference between these two areas can be exercised in the choice of the final optional question.

All steps of the argument or derivation should be written down in those questions which require a given result to be shown. Although less detail may suffice in questions where an unknown value or result is to be found, candidates are advised to show sufficient working so that credit may be earned for their method of solution even if an error occurs.

Non-standard symbols introduced by candidates should be defined or explained, for example by showing forces or velocities in a diagram within their written answers. Annotating a diagram in the question paper is insufficient, since this will not be seen by the Examiners.

General comments

Almost all candidates attempted all the questions, and while very good answers were frequently seen, the paper discriminated well between different levels of ability. In the only question which offered a choice, namely **Question 10**, there was a strong preference for the Statistics option, though some of the candidates who chose the Mechanics option produced good attempts. Indeed all questions were answered well by some candidates, with **Questions 3, 5** (second part) and **9** found to be the most challenging.

Advice to candidates in previous reports to set out their work clearly, with any corrections legible and the replacements to deleted attempts readily identifiable was seemingly heeded by many. When relevant in Mechanics questions it is helpful to include diagrams which show, for example, what forces are acting and also their directions as in **Question 4** and the directions of motion of particles, as in **Question 1**. Where the meaning of symbols introduced by candidates is not clearly defined in this way or is not otherwise obvious, it is helpful to include a definition or explanation. This is also true in Statistics questions, and thus in **Questions 7, 8 and 10** any symbols used when stating the hypotheses should either be in standard notation or should be correctly defined.

The rubric for the paper specifies that non-exact numerical answers be given to 3 significant figures, and while most candidates abided by this requirement, a few rounded intermediate results to this same or lower accuracy with consequent risk that the final result is in error in the third significant figure. Such premature approximations should therefore be avoided, particularly when the difference of two numbers of similar sizes is to be taken, as can happen when taking the difference of means or estimating variances for example.

Comments on specific questions

Question 1

Almost all candidates deduced that the initial impulse causes *A* to move with speed 4 m s^{-1}, and then formulated two equations for the speeds of the spheres after the collision by means of conservation of momentum and Newton's restitution equation. These simultaneous equations were usually solved correctly to give the speeds of *A* and *B* in terms of *e*. For the spheres to move in opposite directions after the collision, *A*'s direction of motion must be reversed in the collision and this yields the required lower bound on *e*.

Answer: $\dfrac{4(2-3e)}{5}$ m s^{-1}, $\dfrac{8(1+e)}{5}$ m s^{-1}.

Question 2

When a question such as this requires that a given result be shown, candidates should set out clearly each step of their proof. Thus it is inadequate to simply state, for example, that the final direction of motion of the sphere P is known to be parallel to its initial direction and then to rely on simple geometry. Ideally candidates should state or at least imply that in each impact with the barrier the component of P's speed along it is unchanged and its component perpendicular to the barrier changes by a factor e. Combining this information about the two impacts yields the required result. This may be done by means of equations, or by showing the various speed components on a comprehensive and clearly labelled diagram. As noted above, anything written by candidates on their question paper is not seen by the Examiners, so any additions they wish to make to the given diagram must be shown by candidates in the work which they hand in, and any symbols introduced for speeds or angles should be defined in some appropriate way. In the second part the loss in kinetic energy is most easily found by subtracting the final from the initial value, perhaps after establishing that P's final speed is eu. Those candidates who attempted to find separately and then add together the losses in the two impacts risked additional and unnecessary working. Many candidates referred to the components of speed along and perpendicular to the barriers as being in a vertical or horizontal direction, mistakenly so since they are all in the plane of the horizontal table, but the Examiners chose to overlook this minor error since the candidates' meaning was usually clear.

Answer: $\frac{1}{2}m\,(1 - e^2)\,u^2$.

Question 3

When attempting such a question concerning simple harmonic motion, it is often advisable to consider carefully whether to relate the displacement x to the time t by $x = a \sin \omega t$ or $x = a \cos \omega t$. While both approaches are valid if employed correctly, one may be more appropriate than the other, involving less work. The deciding factor may well be the value of x when $t = 0$. Turning to the first part of the question, almost all candidates realised (though rarely stated) that the particle next achieves its maximum speed at the centre O. Since the time from M to O is of course equal to the time it would take the particle to move directly from O to M, it follows that $\frac{1}{2}a = a \sin(\omega \times 1)$, and hence $\omega = \frac{\pi}{6}$ and the period T is found from $\frac{2\pi}{\omega}$. The popular alternative of the same equation with sin simply replaced by cos is incorrect, since this represents the time from B to M, and the working must therefore be appropriately modified. In the second part, most candidates successfully equated half the maximum speed, namely $\frac{1}{2}\omega a$, to $\omega\sqrt{(a^2 - x^2)}$, to find the distance x from O. Fortunately for those who did not find the correct value of ω in the first part, the required distance here is independent of ω. In the final part, some initial thought is again advisable as to the optimum method of solution, which is probably to realise that the particle will be $\sqrt{2}$ m from O at a time of 1·5 s after passing through O and hence solve $\sqrt{2} = a \sin(\omega \times 1{\cdot}5)$. It is not of course valid to argue that because 1·5 s is one-eighth of the period then $\sqrt{2}$ must be one-half of the amplitude, since the particle's speed is not constant.

Answer: 12 s; $a\frac{\sqrt{3}}{2}$ m; 2 m.

Question 4

In a similar way, candidates are advised in questions such as this to devote some thought to which moment or force resolution equations will yield the required results most easily. One such approach in the first part is to first find the reaction R_C at C by taking moments for the rod about A. The friction F and reaction R between the cube and the plane are then respectively $R_C \sin 30° = \frac{3\sqrt{3}W}{16}$ and $W + R_C \cos 30° = \frac{25W}{16}$. The required inequality then follows immediately from $F \le \mu R$. If candidates quote instead $F = \mu R$ then they must give some explicit reason for changing from an equality to an inequality. The horizontal component $X = \frac{3\sqrt{3}W}{16}$ and vertical component $Y = \frac{7W}{16}$ of the force at A are found most easily by resolution of forces in their respective directions, and finally the magnitude of the force acting on the rod at A is given by $\sqrt{(X^2 + Y^2}$. If other moment equations are used in the question, then candidates should be careful not to assume (wrongly) that the reaction between the cube and the plane acts at D or at the mid-point of DE.

Answer: $\left(\frac{\sqrt{19}}{8}\right)W$.

Question 5

Finding the required moment of inertia I presented most candidates with little difficulty, requiring the use of standard formulae and the parallel axes theorem to formulate and then sum the individual moments of inertia of the rod, disc and particle. As in all such questions where a given result must be shown, candidates should include sufficient details of their solution so as to justify full credit. Candidates who simply write down a sum of terms with no explanation whatever run considerable risk, since an error in only one term can cast doubt on the validity of their whole process and thereby lose considerable credit. In the second part, the couple acting on the system should be found in terms of $\sin\theta$, where θ is the small angular displacement, and equated to $I\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}$. Approximating $\sin\theta$ by θ yields the familiar form $\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -\omega^2\theta$ of the standard SHM equation. Equating the resulting ω to $\frac{2\pi}{T}$ where T is the given period yields $k^2 - 4k + 3 = 0$, which is readily solved for k. Candidates should be aware that this constant parameter ω in the SHM equation is not equal to $\frac{\mathrm{d}\theta}{\mathrm{d}t}$ even though the same symbol is sometimes used to denote the latter, and thus finding $\frac{\mathrm{d}\theta}{\mathrm{d}t}$ from conservation of energy at some fixed angular displacement ($\frac{\pi}{2}$ was usually chosen) and equating it to the SHM parameter ω is wholly invalid.

Answer: 1 and 3.

Question 6

Most candidates produced good answers to this question. After stating the hypotheses to be tested, a table of the expected reliability of the broadband connection is produced in the usual way and preferably to an accuracy of two (or more) decimal places. The calculated χ^2-value 5·05 (or 5·03 if the expected numbers were found to only one decimal place) should be compared with the tabular value 5·991, leading to acceptance of the null hypothesis. Thus reliability is independent of supplier.

Question 7

The gradient b of the required regression line $x = a + by$ is first found using the standard formula $3{\cdot}25 \times b = 0{\cdot}56^2$, and then used with the means of the sample values of x and y to find the constant term a. These calculations were generally performed well, though candidates are advised to retain at least 3 significant figures in their calculated value of b. Most candidates went on to state the null and alternative hypotheses correctly, which should be in the form $\rho = 0$ and $\rho \neq 0$, though some wrongly stated them in terms of the product moment correlation coefficient r of the sample. This is not the same entity as the coefficient ρ for the population. Comparison of the given value 0·56 of r with the tabular value 0·632 leads to a conclusion of there being no evidence of non-zero correlation.

Answer: $x = 0{\cdot}0965y + 1{\cdot}48$.

Question 8

As in all such tests, the hypotheses required in the first part should be stated in terms of the population mean and not the sample mean. The unbiased estimate 0·3889 of the population variance may be used to calculate a t-value of 1·15. Since it is a one-tail test, comparison with the tabulated value of 1·943 leads to acceptance of the null hypothesis, so the mean distance jumped is not greater than 6·2 m. The quickest way to estimate the population mean μ in the second part of the question is to find the mid-point of the given confidence interval. The unbiased estimate s^2 for the population variance can be found in a broadly similar way from the half-length 0·43 of the same interval, by equating this to $\frac{2.365s}{\sqrt{8}}$. This is of course equivalent to relating 6·75 and 5·89 to $\mu \pm \frac{2.365s}{\sqrt{8}}$ and solving the resulting simultaneous equations, but some candidates would have found the former method quicker and less error-prone.

Answer: 6·32, 0·264.

Question 9

While equating f(2) to unity does appear to show that $a = 1$ this frequently-seen attempt is of course unjustified, and the valid method is to instead recall that the integral of the probability density function f over the whole range $x \geq 2$ is equal to 1. When integrating f(x) to find the distribution function F of X for $x \geq 2$, candidates should remember to include an appropriate constant of integration to ensure that F(2) = 0. For completeness, F(x) should also be stated for values of x less than 2, and saying that it takes the value of 1 "otherwise" should certainly be avoided here and perhaps also in general. The median value m of X was usually found correctly from $\mathrm{F}(m) = \frac{1}{2}$. While most candidates knew how to find or state the cumulative distribution function G of Y and then differentiate it to find the required probability density function g, many did not attempt to simplify the resulting expressions. In the final part P(Y > 10) is readily found by evaluating 1 – G(10).

Answer: 0 (x < 2), $1 - \mathrm{e}^{-(x-2)}$ ($x \geq 2$); 2 + ln 2 or 2·69; (**i**) 0 ($y < \mathrm{e}^2$), $\frac{e^2}{y^2}$ ($y \geq \mathrm{e}^2$); **(ii)** $\frac{e^2}{10}$ or 0·739.

Question 10 (Mechanics)

This optional question was attempted by only a small proportion of the candidates, but many of those who did so made good attempts, particularly at the first part. Applying Newton's second law of motion to the particle P in a radial direction when the string is horizontal yields the tension T in terms of the unknown speed v, and a second equation involving v results from conservation of energy, allowing its elimination to show that T has the required form. The key to the second part is to realise that the string loses contact with the peg when it is vertical, at which point the radius of P's circular motion changes instantaneously from $\frac{a}{2}$ to $\frac{3a}{2}$. The different tensions immediately before and after this instant may be expressed in terms of u as in the first part, and the given information about their ratio enables $u^2 = 5ag$ to be shown. Newton's law and conservation of energy are again applicable in the final part, though the direction of the string, namely $\cos^{-1}\left(\frac{2}{3}\right)$ to the vertical or its equivalent, must also be effectively found and utilised.

Answer: (**ii**) 2 mg.

Question 10 (Statistics)

As in all such tests, the hypotheses should be stated in terms of the population means and not the sample means. The majority of the many candidates attempting this optional question chose to find an unbiased estimate 0·004296 of the combined variance, leading to a calculated z-value of 1·68. Others found instead a pooled estimate 0·1116 of a common variance, though usually without stating explicitly their assumption of a common population variance, and hence a similar calculated z-value of 1·72. Since it is a one-tail test, comparison with the tabulated value of 1·645 leads to acceptance of the alternative hypothesis, namely that the club A cyclists take less time. The second part requires the calculation of a similar z-value but here with a numerator of magnitude 0·11 – 0·05 (not 0·11 + 0·05 as was sometimes seen), giving 0.915 or 0.938 depending on the previous estimate of the variance. Appropriate use of the table of the normal distribution function, coupled with some thought as to the nature of the requirement for α, leads to the set of its possible values.

Answer: $\alpha \geq 18{\cdot}0$ (or 17·4).